The index is less conservative than the condition number of the stiffness matrix and reflects the fact that for some cases, no error magnification occurs even when the condition number is very high. The proposed index was applied to calculation of derivatives of beam response to changes in the beam structural parameters by the semianalytical method. The calculation of derivative with respect to length is very sensitive to errors; whereas the calculation of derivative with respect to cross-sectional height is not. The proposed index indiscriminated well between these two cases.

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Accuracy of Condensed Eigenvalue Solution

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I. Introduction

A LARGE number of degrees of freedom are used to model complex structures. Efficient solution of the resulting eigenvalue problem is of the utmost importance for dynamic analysis. One way of dealing with complex structures is to partition the structure into a number of substructures. Lower natural frequencies and modal vectors of the complete structure can be calculated using the Guyan reduction method. In this case, the boundary nodes of substructures determine the master degrees of freedom for the condensation. The accuracy of the natural frequencies calculated in this manner are determined by the relative magnitude of the natural frequencies of the substructures corresponding to fixed boundary degrees of freedom. The substructure natural frequencies should be very large compared to the global natural frequencies to be calculated; otherwise, the solution of a frequency-dependent eigenvalue problem is necessary in order to obtain natural frequencies accurately. In this Note, a computationally efficient method is presented for the solution of this nonlinear eigenvalue problem.

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Condensation of Stiffness and Mass Matrices

Free vibration analysis of structures results in a matrix equation of the form

$$[K]\{U\} - \lambda[M]\{U\} = \{0\} \tag{1}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices, respectively, $\{U\}$ is the modal vector, and λ is the square of the natural frequency. If this system of equations is partitioned, separating the boundary and internal degrees of freedom, the following equation system is obtained:

$$\begin{bmatrix} K_{BB} & K_{BI} \\ K_{IB} & K_{II} \end{bmatrix} \begin{Bmatrix} U_B \\ U_I \end{Bmatrix} - \lambda \begin{bmatrix} M_{BB} & M_{BI} \\ M_{IB} & M_{II} \end{bmatrix} \begin{Bmatrix} U_B \\ U_I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2}$$

From the second row of these matrix equations,

$$U_I = -(K_{II} - \lambda M_{II})^{-1}(K_{IB} - \lambda M_{IB})U_B \tag{3}$$

Here, we consider the complete solution of the eigenvalue problem in which only the internal degrees of freedom are present:

$$(K_{II} - \lambda M_{II})W = 0 \tag{4}$$

where Λ is a diagonal matrix having substructure eigenvalues on its diagonal, and W is the substructure modal matrix. Using the identities

$$W^T M_{II} W = I \tag{5a}$$

$$W^T K_{II} W = \Lambda \tag{5b}$$

Eq. (3) can be rewritten as

$$U_I = -W(\Lambda - \lambda I)^{-1} W^T (K_{IB} - \lambda M_{IB})U_B \tag{6}$$

where I is the identity matrix.

The first row of Eq. (2) gives

$$(K_{BB} - \lambda M_{BB})U_B + (K_{BI} - \lambda M_{BI})U_I = 0 \tag{7}$$

Equation (6) is substituted in Eq. (7) to give

$$(K_{BB} - \lambda M_{BB})U_B - (K_{BI} - \lambda M_{BI})W(\Lambda - \lambda I)^{-1} W^T (K_{IB} - \lambda M_{IB})U_B = 0 \tag{8}$$

or

$$D(\lambda)U_B = 0$$

Expansion of the second term in Eq. (8) results in the following expression for the dynamic stiffness matrix:

$$D(\lambda) = K_0 - \lambda M_0 - \lambda G(\Lambda - \lambda I)^{-1} G^T \tag{9}$$

where

$$K_0 = K_{BB} - K_{BI}K_{II}^{-1}K_{IB}$$

$$M_0 = M_{BB} + K_{BI}K_{II}^{-1}M_{II}K_{II}^{-1}K_{IB} - K_{BI}K_{II}^{-1}M_{IB} - M_{BI}K_{II}^{-1}K_{IB}$$

$$G = M_{BI}W - K_{BI}W\Lambda^{-1}$$

In Guyan reduction, the last term in Eq. (9) is neglected. Hence, a linear eigenvalue problem is obtained. The effect of this term on the accuracy of the eigenvalues is determined by the magnitude of the diagonal entries

$$\lambda/(\lambda_k - \lambda)$$

where λ_k are the substructure eigenvalues obtained from Eq. (4). It is shown in Ref. 1 that the error in the ith eigenvalue is limited by the following inequality:

$$e_i < (\lambda_i / \lambda_{k\min}) \tag{10}$$

where $\lambda_{k\min}$ is the minimum of the substructure eigenvalues.

If all the substructure modes are included in Eq. (9), then the calculated eigenvalues for the complete structure are exact. However, since substructures are much stiffer than the entire structure, only very few substructure modes are needed in order to achieve reasonable accuracy. Equation (10) can be used as a guide for the desired accuracy. In this case, $\lambda_{k\min}$ should be replaced by the minimum of the substructure eigenvalues which are not included in the analysis.

Solution of the Nonlinear Eigenvalue Problem

There are efficient methods in the literature developed for the solution of large linear eigenvalue problems. However, they cannot be employed for the solution of the nonlinear problem presented by Eq. (9). The eigenvalues of this nonlinear equation system can be isolated using the fact that the determinant of the original dynamic stiffness matrix is zero when it is calculated at one of the system eigenvalues. In order to keep track of the order of the eigenvalues, the Sturm sequence of the original system is also needed. Wittrick and Williams[2] presented the relationship between properties of the original and the condensed equation system. Accordingly, the Sturm sequence of the original system can be formulated as

$$J(\lambda^*) = J_1(\lambda^*) + S\{D(\lambda^*)\} \tag{11}$$

where

$J(\lambda^*)$ = the number of eigenvalues of the original system that are less than a chosen value λ^*

$J_1(\lambda^*)$ = the number of fixed boundary substructure eigenvalues that are less than λ^*

$S\{D(\lambda^*)\}$ = the number of negative entries on the leading diagonal of triangularized $D(\lambda^*)$ in Eq. (9)

For the solution of the original eigenvalue problem in Eq. (2), it is suggested[3] that the Sturm sequence properties of the system be used, as well as the bisection method, to approximate the zero roots of the characteristic equation of the dynamic stiffness matrix in Eq. (9). After that, the eigenvalues and the eigenvectors of the system can be accurately calculated by applying an inverse iteration in the form

$$D(\bar{\lambda})U_B = \lambda' M(\bar{\lambda})U_B \tag{12}$$

where λ is the approximate eigenvalue that can be corrected using λ',

$$\lambda = \bar{\lambda} + \lambda' \tag{13}$$

The equation for the dynamic mass matrix is given[3] as

$$M(\lambda) = -\frac{\partial}{\partial \lambda} D(\lambda) \tag{14}$$

Equation (14) is applied to Eq. (9) to give

$$M(\lambda) = M_0 + G(\Lambda - \lambda I)^{-1}(2\lambda\Lambda - \lambda^2 I)(\Lambda - \lambda I)^{-1} G^T \tag{15}$$

In order to obtain an accurate eigenvalue using Eq. (13), λ' should be small compared to λ. In other words, λ' should be close to the actual root so that $D(\bar{\lambda})$ and $M(\bar{\lambda})$ in Eq. (12) are accurate. This necessitates a large number of bisections. For each trial value of λ^*, the dynamic stiffness matrix has to be decomposed to count the negative entries on the diagonal. The number of bisections can be reduced by using the subpolynomial interpolation. This iteration method is proposed[4] for the solution of the large linear eigenvalue problems but can more efficiently be employed for nonlinear eigenvalue problems.

The idea behind the method is to approximate the characteristic polynomial of the original equation system with a subpolynomial, which is much easier to iterate on, and to use the subpolynomial to approximate the eigenvalue. To obtain the subpolynomial, first, the eigenvalues are isolated using the Sturm sequence property of the system; then, the determinant values obtained during that isolation are used to approximate the value of the determinant at any other point using Lagrange interpolation. If n number of data points ($\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$) are obtained with n decompositions, then the value of the determinant at any point can be approximated by the equation

$$P_0(\lambda) = \sum_{i=1}^{n} L_i P(\lambda_i^*) \tag{16}$$

where P is the characteristic polynomial, P_0 the approximate subpolynomial, and

$$L_i = \prod_{j \neq 1}^{n} (\lambda - \lambda_j)/(\lambda_i - \lambda_j) \tag{17}$$

The determinant of the original dynamic stiffness matrix, $P(\lambda_i^*)$, is related[2] to the determinant of the condensed dynamic stiffness matrix $\det(D(\lambda_i^*))$ as follows:

$$P(\lambda_i^*) = \det(D(\lambda_i^*)) \prod_{k=1}^{n} (\lambda_i^* - \lambda_k) \tag{18}$$

where λ_k are the fixed boundary eigenvalues of the substructures.

Equation (16) can be used to find the approximate roots of the characteristic equation of the original eigenvalue problem in Eq. (2). Then, the eigenvalues and the eigenvectors can be calculated accurately using Eqs. (12) and (13). Finally, the complete eigenvector can be obtained using Eq. (6).

Another Selection for W

The reason for using the substructure modal matrices for W is to transform Eq. (3) into Eq. (6) using Eq. (5). However, Eq. (5) can also be satisfied by the lower triangular matrix resulting from the Cholesky decomposition of the inverse internal mass matrix

$$M_{II}^{-1} = WW^T$$

Although a Λ matrix obtained this way is not diagonal, it does not change the transformation from Eqs. (3) to (6). When Λ is not diagonal, the calculations of the dynamic matrices in Eqs. (9) and (15) are costly. Nevertheless, they may be calculated once for every eigenvalue obtained using Guyan reduction. This way, eigenvalues and eigenvectors are calculated much more accurately using Eqs. (12) and (13). Also, the error due to static condensation is revealed by the relative size of the correction λ' with respect to the eigenvalue λ.

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